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A PROBABILISTIC APPROACH FOR SCATTERING OF LIGHT IN SLAB GEOMETRY-I

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The RAND Corporation
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PREFACE

In this RAND Memorandum, the author presents mathematical results on radiative transfer in slab geometry. This subject has important implications for meteorology, astrophysics, and the detection of nuclear blasts.

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SUMMARY

In this Memorandum a new probabilistic approach to radiative transfer problems is presented in such a way that the integral equations for the stationary and non-stationary scattering functions in a finite inhomogeneous flat layer are derived directly from a somewhat modified form of the Chapman-Kolmogoroff equation.

CONTENTS

A PROBABILISTIC APPROACH FOR SCATTERING OF LIGHT IN SLAB GEOMETRY.I.

1. INTRODUCTION

While the exact theory of radiative transfer in the initial stage has been mainly developed by several authors (Hopf [8], Ambarzumian [1], and Chandrasekhar [7]), the following powerful methods stemmed recently from the principle-of-invariance method: the invariant-imbedding technique due to Bellman and Kalaba [2] (see also Wing [11]), the combined-operations method of Busbridge [5], and the probabilistic method of Sobolev [9] and Ueno [10].

Of the above, the combined-operations method and the probabilistic method do not permit us to derive directly an integral equation for the scattering function in slab geometry, whereas the invariant-imbedding technique does so. In the present paper, a new probabilistic approach for the derivation of the integral equation governing the scattering function in flat layer is elucidated. Even when there is some distribution of emitting source in the medium, we can compute the angular distribution of emergent radiation from the medium by means of the scattering function.

The characteristic feature of the present approach is that an integral equation for the scattering function defined as a truncated generating function of the emission probability is derived directly from the Chapman-Kolmogoroff equation in somewhat modified form. In Sec. 2, we shall

treat the stationary intensity diffusely reflected by a finite inhomogeneous flat layer, and in Sec. 3 the time-dependent diffusely reflected intensity by a finite inhomogeneous flat layer will be considered.

2. THE STATIONARY CASE

Consider a plane-parallel, inhomogeneous, and isotropically scattering atmosphere of finite optical thickness $\tau_1 > 0$, when the lower surface $\tau = \tau_1$ is illuminated by axially symmetric radiation of intrinsic flux F at an angle $\cos^{-1}\mu_o (0 < \mu_o \leq 1)$ with the inward normal to the surface $\tau = \tau_1$.

Let $I(\tau, +\mu)(0 < \mu \leq 1)$ and $I(\tau, -\mu)(0 < \mu \leq 1)$ denote, respectively, the intensities of radiation directed toward the surfaces $\tau = \tau_1$ and $\tau = 0$.

The equation of transfer. The equation of transfer appropriate to the present case takes the form

$$(2.1) \qquad \mu\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega(\tau)}{2} \int_{-1}^{+1} I(\tau, \mu')d\mu',$$

where ω is the albedo for single scattering, i.e., the probability of photon survival, together with the boundary conditions

$$(2.2) \qquad I(0, +\mu) = 0 \quad \text{and} \quad I(\tau_1, -\mu) = \frac{F}{2}\,\delta(\mu - \mu_o) \quad (0 < \mu \leq 1).$$

Then the equation of transfer in the diffuse radiation field is expressed in the form

$$(2.3)\quad \mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega(\tau)}{2} \int_{-1}^{+1} I(\tau, \mu')d\mu' - \frac{F}{4}\omega(\tau)e^{-(\tau_1-\tau)/\mu_0}.$$

Equation (2.3) should be solved subject to the boundary conditions

$$(2.4)\qquad I(0, +\mu) = 0 \quad \text{and} \quad I(\tau_1, -\mu) = 0 \quad (0 < \mu \leq 1).$$

<u>The emission probability distribution</u>. Let $p^*(\mu; \tau, \tau_1)(0 < \mu \leq 1,\ 0 \leq \tau \leq \tau_1)$ denote the probability that a photon absorbed at the level τ will reappear in the direction μ in the radiation emerging from the surface $\tau = \tau_1$.

The integral equation for $p^*(\mu; \tau, \tau_1)$ is written [10] in the form

$$(2.5)\quad p^*(\mu; \tau, \tau_1) = \omega(\tau)[e^{-(\tau_1-\tau)/\mu} + \bar{\Lambda}_\tau\{p^*(\mu; t, \tau_1)\}],$$

where the truncated Hopf operator $\bar{\Lambda}$ is

$$(2.6)\qquad \bar{\Lambda}_\tau\{f(t)\} = \frac{1}{2}\int_0^{\tau_1} f(t)E_1(|t - \tau|)dt.$$

In (2.6), $E_1(t)$ is the first exponential integral defined by

$$(2.7) \qquad E_1(t) = \int_0^1 e^{-t/\mu} \frac{d\mu}{\mu} .$$

In this context it is mentioned that a photon diffusion equation (2.5) is derived [10] from the transfer equation (2.3).

The scattering function. Let the truncated generating function of the emission probability distribution $p^*(\mu;\ \tau,\ \tau_1)$ be denoted by

$$(2.8) \qquad \Pi(\mu/\mu_o,\ \tau_1) = \int_0^{\tau_1} p^*(\mu;\ \tau,\ \tau_1) e^{-(\tau_1-\tau)/\mu_o}\, d\tau .$$

The above function may be considered also as the truncated characteristic function of the emission probability defined in terms of the bilateral Laplace transform.

On the assumption that the truncated generating function $\Pi(\mu \mid \mu_o,\ \tau_1)$ represents also the Markovian conditional probability of diffuse reflection at the lower surface $\tau = \tau_1$, an integral equation for $\Pi(\mu \mid \mu_o,\ \tau_1)$ follows directly from the Chapman-Kolmogoroff equation in somewhat modified form given by

$$(2.9) \qquad \Pi(\mu \mid \mu_o,\ \tau_1 + \Delta) = \int_0^1 \overline{\Pi}(\mu' \mid \mu_o,\ \tau_1) p(\mu \mid \mu';\ \tau_1,\ \Delta) d\mu' ,$$

for $0 < \Delta < \tau_1$, where

$$(2.10) \qquad \overline{\Pi}(\mu \mid \mu_o,\ \tau_1) = \Pi(\mu \mid \mu_o,\ \tau_1)(1 - \frac{\Delta}{\mu_o}) + \Delta \cdot p^*(\mu;\ \tau,\ \tau_1) ,$$

and

$$(2.11) \qquad p(\mu \mid \mu'; \tau_1, \Delta) = \frac{\Delta}{2\mu'} p^*(\mu; \tau_1, \tau_1) + \delta(\mu - \mu')(1 - \frac{\Delta}{\mu'}).$$

In (2.10), the first term on the right-hand side is due to the probability of diffuse reflection of photon by the additional layer Δ without scattering, and the second term represents the probability of photon emergence from the medium with scattering in the additional layer Δ. Equation (2.11) is equivalent [10] to the conditional transition probability used in the derivation of an integral equation for $p^*(\mu; \tau, \tau_1)$.

Letting $\Delta \to 0$ in (2.9), we have

$$(2.12) \qquad \frac{\partial \Pi}{\partial \tau_1} + (\frac{1}{\mu} + \frac{1}{\mu_o})\Pi = p^*(\mu; \tau_1, \tau_1)[1 + \frac{1}{2}\int_0^1 \Pi(\mu' \mid \mu_o, \tau_1)\frac{d\mu'}{\mu'}].$$

From (2.5) we get

$$(2.13) \qquad p^*(\mu; \tau_1, \tau_1) = \omega(\tau_1)[1 + \frac{1}{2}\int_0^1 \Pi(\mu \mid \mu', \tau_1)\frac{d\mu'}{\mu'}].$$

The substitution of (2.13) into (2.12) provides

$$(2.14) \qquad \begin{aligned} \frac{\partial \Pi}{\partial \tau_1} + (\frac{1}{\mu} + \frac{1}{\mu_o})\Pi &= \omega(\tau_1)[1 + \frac{1}{2}\int_0^1 \Pi(\mu' \mid \mu_o, \tau_1)\frac{d\mu'}{\mu'} \\ &+ \frac{1}{2}\int_0^1 \Pi(\mu \mid \mu', \tau_1)\frac{d\mu'}{\mu'} + \frac{1}{4}\int_0^1\int_0^1 \Pi(\mu \mid \mu', \tau_1)\Pi(\mu'' \mid \mu_o, \tau_1)\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''}].\end{aligned}$$

Writing

$$\Pi(\mu \mid \mu_o, \tau_1) = S^*(\tau_1; \mu, \mu_o), \tag{2.15}$$

and using the reciprocity principle given in a preceding paper [10], we get

$$S^*(\tau_1; \mu, \mu_o) = S^*(\tau_1; \mu_o, \mu), \tag{2.16}$$

and (2.14) becomes

$$\frac{\partial S^*}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_o}\right) S^* = \omega(\tau_1) X^*(\mu, \tau_1) X^*(\mu_o, \tau_1), \tag{2.17}$$

where

$$X^*(\mu, \tau_1) = 1 + \frac{1}{2} \int_0^1 S^*(\tau_1; \mu, \mu') \frac{d\mu'}{\mu'} . \tag{2.18}$$

Equation (2.17) is equivalent to that given by Bellman and Kalaba [2], Busbridge [6], and Ueno [10].

Hence the diffusely reflected intensity emergent from the surface $\tau = \tau_1$ is yielded by

$$I(\tau_1, +\mu) = \frac{F}{4\mu} S^*(\tau_1; \mu, \mu_o). \tag{2.19}$$

3. TIME-DEPENDENT CASE

The equation of transfer. Consider a plane-parallel, inhomogeneous, and isotropically scattering atmosphere of finite geometrical thickness $z_1 > 0$ when the lower surface $z = z_1$ is illuminated by parallel rays of time-dependent intrinsic flux $F\delta(t)$, where F is a constant and $\delta(t)$ is the Dirac delta function of time t.

The equation of transfer appropriate to the present case is written in the form

$$(3.1)\quad \mu\frac{\partial I(z, t, \mu)}{\partial z} + \frac{1}{c}\frac{\partial I}{\partial t} + \ell(z)I = \frac{1}{2}\sigma(z)\int_{-1}^{+1} I(z, t, \mu')d\mu',$$

where ℓ and σ are respectively the volume-attenuation and scattering coefficients; c is the speed of light.

Equation (3.1) should be solved subject to the boundary and initial conditions

$$(3.2)\quad I(0, t, +\mu) = 0 \text{ and } I(z_1, t, -\mu) = \frac{F}{2}\,\delta(\mu - \mu_0)\delta(t)$$

$$(0 < \mu \leq 1,\ 0 \leq t),$$

$$(3.3)\quad I(z, 0, +\mu) = 0 \text{ and } I(z, 0, -\mu) = 0$$

$$(0 < \mu \leq 1,\ 0 \leq z \leq z_1).$$

The equation of transfer in the time-dependent diffuse reflection field is given by

$$(3.4)\quad \mu\frac{\partial I}{\partial z} + \frac{1}{c}\frac{\partial I}{\partial t} + \ell(z)I = \frac{1}{2}\sigma(z)\int_{-1}^{+1} I(z,\ t,\ \mu')d\mu' + \frac{F}{4}\sigma(z)e^{-(z_1-z)/\mu_o}\ \delta(t - \frac{z_1-z}{c\mu_o}),$$

together with the boundary and initial conditions

$$(3.5)\quad I(0,\ t,\ +\mu) = 0 \text{ and } I(z,\ t,\ -\mu) = 0 \quad (0 < \mu \le 1,\ 0 \le t),$$

$$(3.6)\quad I(z,\ 0,\ +\mu) = 0 \text{ and } I(z,\ 0,\ -\mu) = 0 \quad (0 < \mu \le 1,\ 0 \le z \le z_1).$$

<u>The emission probability distribution</u>. Let $p^*(\mu;\ z,\ z_1;\ t-s)(0 < \mu \le 1,\ 0 \le z \le z_1,\ 0 \le s < t)$ denote the probability that a photon absorbed at the level z and time s will reappear in the direction μ at time t in the radiation emerging from the surface $z = z_1$. We assume such a medium of dilute gases that the duration of temporal capture of photon is negligible compared with the mean free time. Furthermore, it should be mentioned that the emission probability is considered as homogeneous with respect to time, because of the stationary optical properties of the medium.

The expression for $p^*(\mu;\ z,\ z_1;\ t)$ consists of two terms: The first represents a term of direct transmission without scattering in a time interval t, and the second is due to multiple scattering of photons.

Then we have

$$(3.7)\quad \begin{aligned} p^*(\mu;\ z,\ z_1;\ t) &= \sigma(z)e^{-(z_1-z)/\mu}\,\delta\left(t - \frac{z_1-z}{c\mu}\right) \\ &+ \frac{\sigma(z)}{\alpha}\int_0^{z_1} dz' \int_0^1 p^*\left(\mu; z', z_1; t - \frac{|z-z'|}{c\mu'}\right) e^{-|z-z'|/\mu'}\,\frac{d\mu'}{\mu'} \ . \end{aligned}$$

Putting $z = z_1$, we obtain

$$(3.8)\quad \begin{aligned} p^*(\mu;\ z_1,\ z_1;\ t) &= \sigma(z_1)\delta(t) \\ &+ \frac{\sigma(z_1)}{2}\int_0^{z_1} dz' \int_0^1 p^*\left(\mu; z', z_1; t - \frac{z_1-z'}{c\mu'}\right) e^{-(z_1-z')/\mu'}\,\frac{d\mu'}{\mu'} . \end{aligned}$$

<u>The scattering function</u>. Let the truncated generating function of the emission probability distribution $p^*(\mu;\ z,\ z_1;\ t)$ be denoted by

$$(3.9)\quad \Pi(\mu \mid \mu_o,\ z_1,\ t) = \int_0^{z_1} p^*\left(\mu;\ z,\ z_1;\ t - \frac{z_1-z}{c\mu_o}\right) e^{-(z_1-z)/\mu_o}\,dz .$$

Under the assumption that $\Pi(\mu \mid \mu_o,\ z_1,\ t)$ has the Markovian property of the conditional diffuse reflection probability at the surface $z = z_1$, an integral equation for $\Pi(\mu \mid \mu_o,\ z_1,\ t)$ is directly derived from the Chapman-Kolmogoroff equation in somewhat modified form, namely

$$\Pi(\mu \mid \mu_o,\ z_1 + \Delta,\ t + \frac{\Delta}{c}(\frac{1}{\mu} + \frac{1}{\mu_o})) \tag{3.10}$$
$$= \int_0^t \int_0^1 \bar{\Pi}(\mu' \mid \mu_o,\ z_1,\ S)\, p(\mu \mid \mu';\ z_1,\ t-S,\ \Delta)\, dS\, d\mu',$$

where Δ lies between 0 and z_1,

$$\bar{\Pi}(\mu \mid \mu_o,\ z_1,\ S) = \Pi(\mu \mid \mu_o,\ z_1,\ S)(1 - \frac{\ell(z_1)\Delta}{\mu_o}) \tag{3.11}$$
$$+ \Delta \cdot p^*(\mu;\ z_1,\ z_1;\ s),$$

and

$$p(\mu \mid \mu_o;\ z_1,\ t-s,\ \Delta) = \frac{\Delta}{2\mu_o}\, p^*(\mu;\ z_1,\ z_1;\ t-s) \tag{3.12}$$
$$+ \delta(\mu - \mu_o)\delta(t - s)(1 - \frac{\ell(z_1)\Delta}{\mu_o}).$$

In the limit as $\Delta \to 0$, we get

$$\frac{\partial \Pi}{\partial z_1}(\mu \mid \mu_o,\ z_1,\ t) + (\frac{1}{\mu} + \frac{1}{\mu_o})[\frac{1}{c}\frac{\partial}{\partial t} + \ell(z_1)]\Pi \tag{3.13}$$
$$= p^*(\mu; z_1, z_1; t) + \frac{1}{2}\int_0^t \int_0^1 \Pi(\mu' \mid \mu_o, z_1, s)\, p^*(\mu; z_1, z; t-s)\, ds \frac{d\mu'}{\mu'}.$$

Inserting (3.8) into (3.13), we obtain

$$\frac{\partial \Pi}{\partial z_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)\left[\frac{1}{c}\frac{\partial}{\partial t} + \ell(z)\right]\Pi = \sigma(z)\Big[\delta(t)$$

$$(3.14) \quad + \frac{1}{2}\int_0^1 \Pi(\mu \mid \mu', z_1, t)\frac{d\mu'}{\mu'} + \frac{1}{2}\int_0^1 \Pi(\mu' \mid \mu_o, z, t)\frac{d\mu'}{\mu'}$$

$$+ \frac{1}{4}\int_0^t\int_0^1\int_0^1 \Pi(\mu|\mu', z, s)\Pi(\mu''|\mu_o, z, t-s)ds\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''}\Big].$$

When we write

$$(3.15) \qquad \Pi(\mu \mid \mu_o, z_1, t) = S(z_1, t; \mu, \mu_o),$$

(3.14) becomes

$$\frac{\partial S}{\partial z_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)\left[\frac{1}{c}\frac{\partial}{\partial t} + \ell(z)\right]S$$

$$(3.16) \quad = \sigma(z_1)\Big[\delta(t) + \frac{1}{2}\int_0^1 S(z_1, t; \mu, \mu')\frac{d\mu'}{\mu'} + \frac{1}{2}\int_0^1 S(z_1, t; \mu'', \mu_o)\frac{d\mu''}{\mu''}$$

$$+ \frac{1}{4}\int_0^t\int_0^1\int_0^1 S(z_1, S; \mu, \mu')S(z_1, t-s; \mu'', \mu_o)dS\,\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''}\Big].$$

Equation (3.16) coincides, in the case of isotropic scattering, with equation (21) of [3].

The requisite diffusely reflected intensity at time t is provided by

$$(3.17) \qquad I(z_1, t; +\mu) = \frac{F}{4\mu} S(z_1, t; \mu, \mu_o).$$

<u>The principle of reciprocity</u>. In a manner similar to that used by Biberman and Veklenko [4], we shall derive the reciprocity principle as follows:

Let $f(x, y)$ be the joint frequency function of two one-dimensional variables ξ and η, which is expressed in terms of the corresponding conditional frequency function of $f(x \mid y)$, relative to the hypothesis $\eta = y$, multiplied by the marginal frequency function $f_2(y)$.

Then, according to the reciprocity relation of a random process, we have

$$(3.18) \qquad f(x \mid y) = \frac{f(x, y)}{f_2(y)} = \frac{f(y \mid x) f_1(x)}{f_2(y)} \ .$$

We write

$$(3.19) \qquad S(z_1, t - s; \mu, \mu_o) = S(s, \mu_o; t, \mu),$$

so that $S(s, \mu_o; t, \mu)(0 < \mu \leq 1, 0 < \mu_o \leq 1)$ denotes the probability that a photon absorbed at the surface $z = z_1$ at time s in the direction μ_o will be found at the surface $z = z_1$ at the instant t in the direction μ.

Considering that x-state and y-state variables correspond respectively to μ and μ_o variables, we get

$$(3.20) \qquad S(z_1, t; \mu, \mu_o) = S(z_1, t; \mu_o, \mu)$$

because of the constancy of f_i-function $(i = 1, 2)$ with respect to frequency and angular arguments.

In this context it should be noted that, when redistribution of radiation energy in frequency takes place probabilistically (i.e., noncoherent scattering is either complete or real), the generalized reciprocity principle for noncoherent scattering is formulated with the aid of the principle of detailed balancing.

By use of (3.20), (3.16) reduces to

$$\begin{aligned}(3.21)\qquad &\frac{\partial S}{\partial z_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)\left[\frac{1}{c}\frac{\partial}{\partial t} + \ell(z_1)\right]S \\ &= \sigma(z_1)\int_0^t X(\mu, z_1, s)X(\mu_o, z_1, t - s)ds,\end{aligned}$$

where

$$(3.22)\qquad X(\mu, z_1, t) = \delta(t) + \frac{1}{2}\int_0^1 S(z_1, t; \mu, \mu')\frac{d\mu'}{\mu'} .$$

When the optical properties of the medium are constant throughout the atmosphere, (3.21) becomes

$$\begin{aligned}(3.23)\qquad &\frac{\partial S}{\partial \tau_1} + \left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)\left(\frac{\partial}{\partial u} + 1\right)S \\ &= \omega\int_0^u X(\mu, \tau_1, u')X(\mu_o, \tau_1, u - u')du',\end{aligned}$$

where $\tau = z\ell$, $t_2 = (1/c\ell)$, $u = t/t_2$, $\omega = \sigma/\ell$, and

$$(3.24) \qquad X(\mu, \tau_1, u) = \delta(u) + \frac{1}{2}\int_0^1 S(\tau_1, u; \mu, \mu')\frac{d\mu'}{\mu'} .$$

In the above, t_2 is called the <u>mean free time</u>.

In a later paper, with the aid of the present probabilistic approach, the generalized principle of reciprocity for noncoherent scattering will be treated. Furthermore, the application to the diffuse-reflection problem of radiation in spherical and cylindrical geometries will be made.

REFERENCES

1. Ambarzumian, V. A., "Theoretical Astrophysics," Pergamon Press, New York, 1958.

2. Bellman, R., and R. Ka aba, "On the principle of invariant imbeddir. ind propagation through inhomogeneous media," Pro. Nat. Acad. Sci. USA., Vol. 42 (1956), pp. 629-632.

3. Bellman, R., R. Kalaba, and S. Ueno, "Invariant imbedding and time-dependent diffuse reflection by a finite inhomogeneous atmosphere," ICARUS, Vol. 1 (1962), pp. 191-199.

4. Biberman, L. M., and B. A. Veklenko, "Generalized reciprocity principle," J. Expel. Theoret. Phys. (USSR), Vol. 39 (1960), pp. 88-93.

5. Busbridge, Ida W., "The Mathematics of Radiative Transfer," Cambridge Tract, No. 50, Cambridge University Press, London, 1960.

6. _______________, "On inhomogeneous stellar atmospheres," Astrophys. J., Vol. 133 (1961), pp. 198-209.

7. Chandrasekhar, S., "Radiative Transfer," Oxford University Press, London, 1950.

8. Hopf, E., "Mathematical Problems of Radiative Equilibrium," Cambridge Tract, No. 31, Cambridge University Press, London, 1934.

9. Sobolev, V. V., "Transfer of Radiation Energy in the Atmospheres of Stars and Planets," Moscow, 1956.

10. Ueno, S., "The probabilistic method for problems of radiative transfer. XIII. Diffusion Matrix," J. Math. Anal. Appl., Vol. 4 (1962), pp. 21-37.

11. Wing, Milton G., "An Introduction to Transport Theory," John Wiley & Sons, New York, 1962.

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